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1986 J. Phys. A: Math. Gen. 19 L373

(http://iopscience.iop.org/0305-4470/19/7/002)

J. Phys. A: Math. Gen. 19 (1986) L373-L380. Printed in Great Britain

LETTER TO THE EDITOR

Defective degenerate mode and dynamics of perturbed sine-Gordon soliton plus phonon

P C Dash

Department of Physics, College of Basic Science and Humanities, Orissa University of Agriculture and Technology, Bhubaneswar-751 003, India

Received 17 July 1985

Anstract. Two new ideas are introduced in the case of the sine-Gordon 2π kink. The first refers to non-interacting phonon plus soliton which is found to be the correct picture in first-order theory. This justifies the additive paradigm of soliton plus oscillation plus translation and hence the Newtonian dynamics. The non-Newtonian behaviour is due to an inappropriate consideration of the soliton-phonon wavepacket interaction. The significance of the defective degenerate mode to the sG 2π kink in the presence of perturbing force fields and/or damping is discussed and a surprising result (the absence of the mode) is reported.

A complete set of functions used in linear stability analysis of non-linear Klein-Gordon kinks is very helpful in discussing the effect of small perturbations (Fogel *et al* 1976, 1977) in addition to its usual role in quantum field theory (Goldstone and Jackiw 1975) and low-temperature kink statistical mechanics (Krumhansl and Schrieffer 1975). Very recently the same complete set has attracted much attention due to its appearance in the controversial non-Newtonian dynamics (Reinisch and Fernandez 1981) and has also explained a new defective degenerate mode (Magyari and Thomas 1984). In linear stability analysis as well as perturbation calculations a small space- and time-dependent function is expanded in terms of this complete set of spatial functions and the time-dependent expansion coefficients describe the time development of the problem. In a recent letter I have given a good account of the Newtonian/non-Newtonian controversy (Dash 1983, 1985 and references therein). There, as well as in another paper (Dash 1984), I have stressed the importance of initial conditions on sG 2π kink solitons; in particular I attempted to show that the non-Newtonian dynamics is due to the specific initial values chosen by Reinisch and Fernandez (1981) (RF). A general set of initial conditions was taken to display a different dynamics of the 2π kink which is dependent upon a characteristic time t_0 (near this time the soliton receives a Newtonian acceleration and at times far away from t_0 the soliton moves with a constant velocity accompanied by a shape variation). When t_0 goes to zero these initial values become identical to those of RF and the dynamics is in complete accord with their predictions. In an attempt to explain these results qualitatively I assumed that when a soliton is at rest with phonons to begin with, the interaction takes place in the presence of imperfections. This interaction I conjectured to be dependent upon initial conditions. Now many important questions remain unanswered: how will this feature of soliton plus phonon be brought into the domain of linear perturbation theory? Once mathematically introduced, how will they develop in time? Why does there exist so much controversy regarding this Newtonian/non-Newtonian behaviour? Just by choosing the integration constant associated with scattering states in the form of a delta function I introduce a very novel expression which I define as phonon plus soliton and I intend to show that it is the only appropriate structure in first-order perturbation theory. This will at once answer all the questions raised and will also explain why the dynamics would always be Newtonian in the presence of small perturbations.

The other topic I want to analyse is the role of the defective degenerate mode (DDM) in studying soliton perturbation. Magyari and Thomas (1984) (MT) propose this mathematical time-degenerate mode in the case of a pure undamped sine-Gordon (SG) equation as well as in the presence of a viscous damping term. Previously they have discussed a similar universal inertia mode in the presence of a force field (which is less than a maximum value and is not perturbative). In the first two cases they discover defective degeneracy (DD) around a moving and a static 2π kink respectively. In the last case it is around a solution of the full inhomogeneous forced SG equation which does not exist in closed form. Their approach is the linear stability analysis which exposes the existence of the DD. They have found applications of this mode in many physical systems such as one-dimensional magnets, the Josephson transmission line, the Trullinger–Bishop multi-component field model, nematic liquids and spin dynamics in ^{3}He. My purpose here is to study the DD around a 2π kink solution which exists analytically in the free and damped case as well as with or without a perturbing force field. After identifying it my next task is to examine the role it plays in the case of constant and periodic force field. Thus my aim is completely different from MT who found DD in the case for homogenous equations always, whereas I search for its significance for inhomogeneous perturbed SG equations. The result that I am going to report is very surprising in the sense that it happens to be negative and so I perforce advance a revised interpretation. The DDM does not represent anything separate, but is only a part of the Goldstone mode (or the specific scattering mode).

To be more specific about the meaning of soliton plus phonon, let us re-examine the origin and interpretation of this complete set of functions. I consider here a solution

$$\theta = S_v(z) + u(z, t) \tag{1}$$

of the sine-Gordon equation

$$\theta_{tt} - \theta_{zz} + \sin\theta = 0 \tag{2}$$

where subscripts t and x represent partial derivatives, z is the coordinate in the soliton rest frame $S_v(z) = 4\tan^{-1}\exp(z)$ and $u(z, t)$ is a small fluctuation field. Linearising in terms of u

$$u_{tt} - u_{zz} + (1 - 2\,\mathrm{sech}^2 z)u = 0. \tag{3}$$

Now u can be expanded as

$$u(z, t) = a_b(t)f_b(z) + \int_{-\infty}^{+\infty} \mathrm{d}k\, a_k(t)f_k(z) \tag{4}$$

where $\{f_b, f_k\}$ represents a complete set of functions

$$p_b = 0 \qquad f_b(z) = (1/\sqrt{2})\,\mathrm{sech}\, z$$

and

$$p_k^2 = 1 + k^2 \qquad f_k(z) = (1/\sqrt{2\pi})\exp(\mathrm{i}kz)(k + \mathrm{i}\tanh z)/p_k.$$

Putting back expression (4) in (3), multiplying by $f_b(z)$ or $f_k^*(z)$ and then integrating over z we obtain, in first order, independent equations for coefficients $a_b(t)$ and $a_k(t)$. After substituting the solutions of these equations in (4), $u(z, t)$ can be expressed in either of the following forms:

$$u(z,t) = (Mt+N)f_b(z) + \int_{-\infty}^{+\infty} \mathrm{d}k(A(k)\exp(-\mathrm{i}p_k t) + B(k)\exp(\mathrm{i}p_k t))f_k(z) \qquad (5)$$

or

$$u(z,t) = (Mt+N)f_b(z) + \int_{-\infty}^{+\infty} \mathrm{d}k(C(k)\cos p_k t + D(k)\sin p_k t)f_k(z) \qquad (6)$$

where M, N, $A(k)$, $B(k)$ or $C(k)$, $D(k)$ are integration constants. The discussion of different aspects of (5) is essential so far as the contents of this letter are concerned. Firstly, when $A(k) = B(k) = 0$, $u(z, t)$ in this homogeneous case means a soliton translation only, i.e. all internal perturbational energy can be used up in giving the soliton a motion with constant velocity. Secondly, if $M \neq N \neq 0$ and $C(k) = (Ak/\sqrt{\pi}p_k)$, $B(k) = 0$, then $u(z, t) = (Mt + N - A)f_b(z)$; the soliton receives a displacement in the negative z direction due to the choice of this phonon wavepacket, or considering another type of wavepacket given by $C(k) = A\pi/(\sqrt{\pi}\sqrt{2}p_k^3 \sinh \frac{1}{2}\pi k)$ (this choice will later be shown to be responsible for the so-called non-Newtonian behaviour of sG kinks)

$$\begin{aligned} u(z,t) = {} & (Mt+N)f_b(z) - (\pi A/2\sqrt{2})t^2 f_b(z) - A\cos t \tanh|z| \\ & + (\pi A/4)[|z|\mathrm{e}^{-|z|} + \tanh|z|\mathrm{e}^{-|z|}(1+|z|)] \\ & + (A/2)\sum_{n=1}^{\infty} 4(-1)^n \cos(1-4n^2)^{1/2} t\, \mathrm{e}^{-2n|z|}(2n + \tanh|z|)/(4n^2-1)^2 \ldots \end{aligned} \qquad (7)$$

This means the soliton receives an acceleration due to the particular choice of $C(k)$, even in the absence of any external force field. After all these discussions regarding the effect of k-dependent coefficients let us consider a choice $B(k) = A(k) = \Sigma_i\, \delta(k - k_i)$ or for simplicity $A(k) = \Sigma_i\, \delta(k - k_i)$, $B(k) = 0$, then equation (7) means uniform motion of a soliton (dependeng on M and N values) in a uniformly moving phonon field. This state I define as soliton plus phonons: at $t = 0$, the scattering functions describe a static small oscillation phonon field and as time advances it becomes a travelling phonon field.

Further examination of (7) shows that a term $t\, f_b(x)$ occurs along with constant M. So if $M \neq 0$ and is small, then for small t values it imparts a small velocity to static solitons. As (3) is a homogeneous equation each and every term of expression (7) represents an independent solution and this independent solution is regarded as a defective degenerate mode by Magyari and Thomas. From our discussions it follows that so long as $u(z, t)$ is small this does not represent a separate mode, but rather a manifestation of the time-dependent coefficient of the Goldstone zero-frequency translation mode.

Next I shall consider the effect of a constant external perturbing force field

$$u_{tt} - u_{zz} + (1 - 2\,\mathrm{sech}^2\, z)u = E. \qquad (8)$$

Obtaining solutions to (8) in analytic form Fogel *et al* (1977) predicted a Newtonian particle-like behaviour of sG kinks which was challenged very recently by Reinisch and Fernandez (1981, 1982). At present many authors advance arguments in favour of Newtonian dynamics thereby refuting the arguments of RF by different considerations (Dash 1985). But the mathematical analysis and numerical simulation of RF still gives the impression that the dynamics is in general non-Newtonian although for some limited specific cases it may be Newtonian. Another motivation for the examination of this case stems from the fact that the work of Kosevich and Kivishar (1984) stresses the Newtonian dynamics of sG kinks from the standpoint of inverse scattering theory except in the case of some odd perturbations which include the constant force field and proposes non-Newtonian behaviour for these odd perturbations. Further, my purpose here is also to illustrate the importance, if any, of the defective degenerate mode in the presence of small perturbations with an example where the integrals involved can be exactly evaluated. Proceeding in a similar manner as indicated previously, (8) yields

$$u(z,t)=[Mt+(\pi E/2\sqrt{2})t^2+N]f_b(z)+\mathrm{d}k(C(k)\cos p_k t+D(k)\sin p_k t+F(k))f_k(z) \tag{9}$$

with

$$F(k)=(2\pi k\delta(k)-\pi/\sinh\tfrac{1}{2}\pi k)E/(\sqrt{2\pi}p_k^3).$$

The RF initial conditions make $M=N=0$ and $C(k)=-F(k)$. As I have pointed out earlier with this choice, even without the presence of a force the kink receives a negative acceleration. So the cancellation of the acceleration term is not an effect of an external force field but it is the behaviour of the particular wavepacket chosen. In some of my earlier papers I have worked out the effect of choosing different values for $C(k)$ and $D(k)$ and obtained a different dynamics of the 2π kink (Dash 1985). Now to identify where the considerations of RF are misleading, let us discuss the dynamics of phonon plus solition, i.e. $C(k)=\Sigma_i\,\delta(k-k_i)$ and $D(k)=\Sigma_i\,\delta(k-k_i)$ and $M=N=0$; then

$$u(z,t)=(\pi E/2\sqrt{2})t^2 f_b(z)+J(z)$$

$$+\text{small amplitude phonon waves travelling in both directions}$$

where

$$J(z)=E\Bigg[\tanh|z|\bigg(1-\tfrac{1}{4}\pi\,\mathrm{e}^{-|z|}+2\sum_{n=1}^{\infty}(-1)^n\exp(-2n|z|)/(4n^2-1)^2$$
$$-(\pi/4)|z|\exp(-|z|)\bigg)$$
$$+4\sum_{n=1}^{\infty}(-1)^n n\exp(-2n|z|)/(4n^2-1)^2-(\pi/4)|z|\exp(-|z|)$$

and represents soliton shape variation. So due to the perturbing field E, the soliton receives an acceleration proportional to the force and a shape variation $J(z)$; all particles in the medium undergo small oscillations constituting travelling phonon waves which are usual linear phonons only suffering a phase change near the soliton. The defective degenerate mode has no special role to play. It appears along with integration constant M and is only a part of the time-dependent coefficient of the Goldstone translation mode. $Mtf_b(x)$ is not an independent solution in this inhomogeneous case.

Next let us consider the sine-Gordon model with damping and a constant force (the damping and force are considered to be perturbations to the pure sG 2π kink):

$$\theta_{tt}-\theta_{zz}+h\theta_t+\sin\theta=E \tag{10}$$

h being the damping coefficient. The equation for the perturbation function $u(z, t)$ in the soliton rest frame can be solved to yield

$$u(z, t) = [M \exp(-2gt) + N + Ft/2g] f_b(z) + J(z) + \int_{-\infty}^{+\infty} \mathrm{d}k \exp(-gt)(C(k) \cos rt + D(k) \sin rt) \quad (11)$$

where $2g = h(1-v^2)^{1/2}$, $F = 4\sqrt{2}vg + (\pi/\sqrt{2})E$, and $2r = (4p_k^2 - h^2)^{1/2}$. Since the damping coefficient $h \ll 1$ and $p_k^2 (= 1 + k^2)$ is always greater than 1, no other case except $p_k > h$ occurs and hence the DDM does not arise for $k \neq 0$. Inspection of the coefficient of $f_b(z)$ shows that though $tf_b(z)$ appears with M and F it has no specific significance except as a time-dependent coefficient of $f_b(z)$. (MT does not propose DD for this case though from the calculations I find a $tf_b(z)$ term.) Again the choice of $C(k)$ and $D(k)$ will determine the dynamics and the soliton plus phonon will obey Newtonian dynamics and shape variation governed by $J(z)$.

Finally, I shall consider the case where DD is expected in $k \neq 0$ mode:

$$\theta_{tt} - \theta_{xx} + h\theta_t + \sin\theta = E \cos pt. \quad (12)$$

Here h may not be small but the right-hand side represents a periodic perturbing force field. With the RHS equal to zero, (12) has a solution—the static kink solution of the pure sG equation. However in this case MT predicted the existence of a DD mode for k values corresponding to $h^2 = 4p_k^2$. Considering $u(x, t)$ as a perturbation function in the soliton rest frame (here the laboratory frame) the coefficients $a_b(t)$ and $a_k(t)$ satisfying the following equations:

$$a_{btt} + ha_{bt} = (\pi/\sqrt{2})E \cos pt \quad (13)$$

$$a_{ktt} + p_k^2 a_k + ha_{kt} = F(k) \cos pt \quad (14)$$

with solutions

$$a_b(t) = M + N \exp(-ht) + (hE/p) \sin pt/(p^2 + h^2) - E \cos pt/(p^2 + h^2)$$

$$a_k^{(1)}(t) = (A(k) \exp qt + B(k) \exp(-qt)) \exp(-ht/2)$$

for $h^2 > 4p_k^2$ with $2q = (h^2 - 4p_k^2)^{1/2}$

$$a_k^{(2)}(t) = (p + Qt) \exp(-ht/2) \qquad \text{for } h^2 = 4p_k^2$$

and for $h^2 < 4p_k^2$ with $(4p_k^2 - h^2)^{1/2} = 2r$

$$a_k^{(3)}(t) = (C(k) \cos rt + D(k) \sin rt) \exp(-ht/2)$$

so that

$$u(x, t) = a_b(t) f_b(x) + \int_{-\infty}^{+\infty} H(k) f_k(x)\, \mathrm{d}k + \mathrm{i}(P + Qt)(\sin mx + \tan hx \cos mx) 4 \exp(-ht/2)/(\sqrt{2\pi}h) + \lim_{\varepsilon \to 0} \int_{-\infty}^{-m-\varepsilon} a_k^{(3)} f_k(x)\, \mathrm{d}k + \int_{m+\varepsilon}^{+\infty} a_k^{(3)} f_k(x)\, \mathrm{d}k + \int_{-m+\varepsilon}^{m-\varepsilon} a_k^{(1)} f_k(x)\, \mathrm{d}k \quad (15)$$

with

$$H(k) = (hp \sin pt + (p_k^2 - p^2) \cos pt)/[(p_k^2 - p^2) + h^2 p^2]$$

and

$$m = \tfrac{1}{2}(h^2 - 4)^{1/2}.$$

For the steady state

$$u(x, t) = [N \exp(-ht) + (hE/p) \sin pt/(p^2 + h^2) - E \cos pt/(p^2 + h^2) + M]f_b(x) + \int_{-\infty}^{+\infty} H(k) f_k(x)\, dx. \qquad (16)$$

The defective degenerate mode as discussed by MT is $t \exp(-ht/2) f_k(x)$ at $k = \pm\tfrac{1}{2}(h^2 - 4)^{1/2}$ or $4p_k^2 = h^2$. Its contribution is the presence of a perturbing force field is the factor associated with the integration constant Q in (15) and zero in the steady state equation (16).

To summarise, I find that equation (4) contains all information regarding the perturbation. For the first time I bring out two distinct characteristics (with or without small external force fields and/or damping): for a particular delta function choice of $A(k)$ and $B(k)$, the effect of perturbation reduces to a picture of a soliton in a sea of non-interacting phonons and as time goes on this non-interacting nature is preserved but the phonons now represent travelling waves instead of being static, and the soliton follows Newtonian dynamics accompanied by small deformations. Any other choice of these integration constants (which may be introduced by a different choice of initial conditions) makes these phonon waves interact and the picture that evolves is a soliton interacting with a phonon wavepacket. Without any external field or damping one trivial choice of $A(k)$ and $B(k)$ gives us $u(z, t) = (Mt + N - A) f_b(z)$; a non-trivial choice yields equation (7). In the presence of a force field (that is, from equation (9)) one arrives at the RF non-Newtonian behaviour by the choice of initial conditions as in our non-trivial example (equation (7)). In all these cases a single feature becomes evident—the choice of integration constants introduces a pole at $k = i$, so that, after the k integration a term equivalent to negative translation mode appears (the integration is nothing more than the representation of phonon interaction). The interaction due to the particular choice of k $(= i)$ gives rise to a soliton translation. Here a question may arise as to how this translation mode, which is orthogonal to the scattering functions, appears as a result of these functions. It is not the scattering functions which generate the bound state; rather, the integration does, which means that it is the interaction of these scattering functions which is responsible for this mode (it is quite evident in the completeness relation itself). However one inconsistency is involved here: when we have neglected higher-order terms in field variables we have eliminated phonon-self and phonon-soliton interactions (Hasenfratz and Klein 1977, Wada and Schrieffer 1978). Now the choice of integration constants brings in a particular type of interaction involving a pole at $k = i$. In conclusion, I propose that in first-order linear theory the appropriate picture should be that of soliton plus phonon in the sense introduced in this paper, where the soliton always receives an acceleration proportional to the force and undergoes a small variation of shape.

As far as the defective degenerate mode is concerned my purpose here is completely different from that of Magyari and Thomas. For the three cases (8), (10) and (12) I consider E, h and $E \cos pt$ as perturbations around a moving SG 2π kink (for equation (8)) or around a static 2π kink (in the cases described by (10) and (12)). For equation (10) Magyari (1984) discusses an inertia mode around $\theta_s(x, t)$ where $\theta_s(x, t)$ is a solitary wave solution of the full equation (10) and E is less than a maximum value E_m (E is not a perturbation and $\theta_s(x, t)$ does not exist in closed form). However with $E=0$, equations (9) and (10) possess DDM $tf_b(x)$ and $t\exp(-ht/2)f_k(x)$ (at $4p_k^2=h^2$) respectively. Hence when $E \neq 0$ I analyse the role of DD in describing the effect of perturbation on the 2π kink; since it is not a degeneracy in space but in time it has nothing to do with the complete set of functions; rather, it appears in the time-dependent expansion coefficients. After a thorough investigation of the perturbed equation the results found are surprising: it is associated with integration constants M and Q, it does not exist in the steady state equation (16), and furthermore it is found in situations where it should not occur (equation (11)). Usually when an initial static kink is considered, $M=0$ and the $tf_b(x)$ term disappears. On the whole I come to the conclusion that DD does not represent a separate mode and the interpretation of MT needs modification. Thus my investigation shows a negative result: defective degenerate mode has no specific role to play in the presence of constant/time-dependent perturbing force fields with or without damping. It is only a part of the Goldstone mode (or specific scattering state function). Hence in accordance with what was found for the perturbed sine-Gordon equation I advance a new interpretation of the DDM of Magyari and Thomas by saying that in their analysis nothing is lost if it is considered as a part of the Goldstone mode (or specific scattering state) occurring through the time-dependent coefficients.

Now I wish to discuss one limitation of the present perturbation method: the appearance of secular terms in expressions (7), (9), (5) and (11). In particular, the presence of t^2 terms in (7) and (9) limits the validity of the theory to the $t=E^{1/2}$ value. In other cases (5) and (11) $tf_b(z)$ is the only secular term. This latter factor is usually avoided with the choice of a motionless initial kink in the Lorentz rest frame (the soliton rest frame). But the former t^2-dependent term can be removed by special methods where adiabatic kink motion is to be assumed (McLaughlin and Scott 1978). The conclusions that have been arrived at for $t \sim E^{1/2}$ would hold good in all probability when this secularity is removed. In expressions (15) and (16) which contain a periodic perturbing force as well as damping, these secular terms are absent; so the results presented here are valid for all time values in many cases where as in certain specific examples (for constant force field) the appropriateness of our discussion is limited for small time values due to asymptotically increasing time-dependent terms. Once this secularity is removed in those few ccases it is hoped that the same conclusions will follow.

Finally, a comparison of the present method with other existing perturbation theories deserves consideration. The most elaborate of all is the one which depends heavily on the inverse scattering method, but the results obtained by Kosevich and Kivshar (1983) differs significantly from Karpman and Solov'ev (1981). The other, more physical, approach is that of McLaughlin and Scott (1978). With the help of both these methods not only the single soliton case but the general multi-soliton perturbation can be handled in principle (even though mathematical calculations are very complicated). However since in the former case there is a discrepancy in the two different approaches and a questionable adiabatic assumption in the latter, then in order to check the results another more simple, more physical and easily accessible approach is needed. Such an approach is possible with the help of a complete set of functions associated with linear stability analysis. But as is shown here one should be extra cautious: the choice of expansion coefficients a_b and a_k are highly significant. If a_k represents a k-dependent expression without involving a pole at $k=\pm i$, the motion of the soliton is accompanied by a phonon wavepacket; if it involves a pole then this phonon cloud has some dynamical effect on the soliton motion. On the other hand the choice of a_k in the form of a delta function represents a soliton accompanied by a non-interacting phonon field which I have described here as 'soliton plus phonon'.

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